M. A. Thesis

SUBHARMONIC RESONANCE

by

John E. Mulhern, Jr.

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INTRODUCTION

There is a great deal of current interest in nonlinear phenomena. The mathematics involved in describing such systems is considerably more complicated than if one is dealing with linear systems. This paper is a survey of various methods of solving differential equations which arise in connection with nonlinear vibrating systems. No exact method of solution has been worked out, but many approximate schemes are in existence. An attempt is made to state as clearly as possible the mathematical techniques that are involved, so that the reader may readily apply them to other situations, as his interests direct. Four of the more common developments are considered in this paper.

Dr. P. O. Pedersen worked out a method whereby a Fourier Series is assumed as a solution to the differential equation which describes certain nonlinear vibrating systems. Since the Fourier series is assumed to be a solution, it should give rise to an identity when substituted into the differential equation. When one makes use of this condition, and assumes that the oscillations are small and the systems are not extremely nonlinear, a set of equations are obtained which give sufficient data to calculate the coefficients of

the first few terms of the series. This gives an approximate solution for small oscillations.

For systems which lead to such complicated equations that numerical analysis procedures are the only practical avenue of attack, the method of differences may be used. The advantage of this method is that the final result will be a function rather than just a number. The dependent variable must be calculated for several values of the independent variable in the region of the curve which one is working. From this data all derivations may be calculated and a Taylor's series formed. When this Taylor's series is transformed into a Fourier series the solution will be in a form in which subharmonic resonance may be investigated.

Another powerful tool in attacking this type of problem is the LaPlace transform. This is the first method to use initial condition. Here small nonlinearity is assumed and if "a" is the coefficient of nonlinearity, the solution is assumed in series form and developed in powers of "a". Each approximation is complete within itself, and better and better results are attained when higher approximations are used. L. A. Pipes uses three approximations for engineering work. However as "a" is not very small compared to unity, more terms must be used in order to attain the desired accuracy.

A similar type of solution may be obtained without the use of the LaPlace transform. The iteration method considers a reasonably general nonlinear differential equation and shows that its solution will contain a subharmonic term. Here initial conditions are used. This is not the most general type of equation, so that it cannot be assumed that because the equation is nonlinear subharmonic resonance will be present. However, in all cases that I have considered a subharmonic term will be present.

These methods form a crossection of the more elementary developments in this field. Application of these methods may be made to many practical vibrating systems, both mechanical and electrical. An example of such a system would be found in the design problems involved in the construction of loudspeakers.

A thorough understanding of the ideas and techniques involved in this paper will aid the reader in undertaking more advanced treatments in this and allied fields.

Subharmonics in Forced Oscillations in Dissipative Systems

The following equation affords the solutions to all systems, with one degree of freedom.

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + sx = F_0 \sin 2wt$$

Where we assume

$$m = m_0 (1 + \mu_1 x + \mu_2 x^2 + \dots)$$
$$r = r_0 (1 + \beta_1 x + \beta_2 x^2 + \dots)$$
$$s = s_0 (1 + \alpha_1 x + \alpha_2 x^2 + \dots)$$
$$F = F_0 (1 + g_1 x + g_2 x^2 + \dots)$$

Assume (1) small amplitudes

$$(2)\ x = \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt$$

((a)) all conditions of convergence are met.

Consider the case of a linearly variable force:

The equation would be,

$$m_0 \frac{d^2x}{dt^2} + r_0 \frac{dx}{dt} + s_0 x = F_0 (1 + g_1 x) \sin 2wt.$$

The frequency of the driving force is 2 $f = 2w$

$$f = \frac{w}{\pi}$$

let

$$2\gamma = \frac{r_0}{m_0} \qquad G_0 = \frac{F_0}{m_0} \qquad h_0 = \frac{g_1 G_0}{2w_0^2}$$

$$w_0^2 = \frac{s_0}{m_0} \qquad P_0 = \frac{\gamma}{w_0} \qquad \xi = \frac{w}{w_0}$$

What conditions must be met among the coefficients so that

$$x = \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt$$

is a solution?

Differentiation may be performed inside the summation. Thus

$$\frac{dx}{dt} = \sum_{n=1}^{\infty} nw \left[A_n \cos nwt - B_n \sin nwt\right]$$

$$\frac{d^2x}{dt^2} = \sum_{n=1}^{\infty} -n^2w^2 \left[A_n \sin nwt \quad B \cos nwt\right]$$

Substituting these results into the differential equation gives:

$$\frac{B_0 w_0^2}{2} + \sum_{n=1}^{\infty} \left\{ \left[(w_0^2 - n^2w^2) A_n - 2\gamma nw B_n\right] \sin nwt \right.$$

$$+ \left[(w_0^2 - n^2w^2) B_n + 2\gamma nw A_n\right] \cos nwt = \left[G_0 \sin 2wt\right]$$

$$\left[1 + g_1 \left(\frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right)\right]$$

It remains to put the right side of the equation into a more tractable form. This can be done by using the formulas:

$$\sin\alpha \sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\sin\alpha \cos\beta = \tfrac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

let

$$\alpha = 2wt$$

$$\beta = nwt$$

thus

$$A_n \sin 2wt \sin nwt - \frac{A}{2} n \left[\cos(2 - n)wt - \cos(2 + n)wt\right]$$

and

$$B_n \sin 2w \cos nwt = \frac{Bn}{2}\left[\sin (2-n)\, wt + \sin (2+n)\, wt\right]$$

thus the right side may be written

$$\left[G_0 + \frac{G_0 g_1 B_0}{2}\right]\sin 2wt + G_0 g_1 \sum_{n=1}^{\infty} \frac{An}{2}\left[\cos (2-n)\, wt - \cos(2+n)wt\right]$$
$$\frac{Bn}{2}\left[\sin (2-n)\, wt + \sin(n+2)wt\right]$$

Divide the whole equation through by w_0^2 and substitute h_0, ξ, and P_0 for appropriate quantities.

$$\frac{B_0}{2} + \sum_{n=1}^{\infty} \left[(1 - n^2\xi^2)A_n - 2P_0 \xi\, n\, B_n\right]\sin nwt + \left[(1-n^2\xi^2)B_n + 2nP_0\xi A_n\right]$$

$$\cos nwt = \left[\frac{G_0}{w_0^2} + h_0 B_0\right] \sin 2wt + h_0 \sum_{n=1}^{\infty} A_n\left[\cos(n-2)wt - \cos(n+2)wt\right]$$

$$B_n\left[\sin (n+2) - \sin (n-2)wt\right] \quad .$$

Since this is to be an identity demand that the coefficients of like terms be equal. The following system of equations results:

$$\frac{B_0}{2} - h_0 A_2 = 0$$

$$(1 - \xi^2)\, A_1 - 2p_0 \xi B_1 - h_0 (B_1 - B_3) = 0$$
$$(1 - \xi^2)\, B_1 + 2p_0 \xi A_1 - h_0 (A_1 + A_3) = 0$$

$$(1 - 4\xi^2)\, A_2 - 4p_0 \xi B_2 - h_0 (2B_0 - B_4) = G_0/w_0^2$$
$$(1 - 4\xi^2)\, B_2 + 4p_0 \xi A_2 - h_0 A_4 = 0$$

$$(1 - 9\xi^2)\, A_3 - \underline{6p_0\xi} B_3 - h_0\,(B_1 - B_5) = 0$$

$$(1 - 9\xi^2)\, B_3 + 6p_0\xi A_3 + h_0\,(A_1 - A_5) = 0$$

$$(1 - 16\xi^2)\, A_4 - 8p_0\xi B_4 - h_0\,(B_2 - B_6) = 0$$

$$(1 - 16\xi^2)\, B_4 + 8p_0\xi A_4 + h_0\,(A_2 - A_6) = 0$$

etc.

Note that in any one equation only even subscripts or only odd subscripts appear.

Thus the above system of equations may be split into two sets of equations in the following manner:

Set I:

$$\tfrac{1}{2}B_0 - h_0 A_2 = 0$$

$$(1 - 4\xi^2)A_2 - 4p_0\xi B_2 - h_0(2B_0 - B_4) = G_0/w_0^2$$

$$(1 - 4\xi^2)B_2 + 4p_0\xi A_2 - h_0 A_4 = 0$$

$$(1 - 16\xi^2)A_4 - 8p_0\xi B_4 - h_0(B_2 - B_6) = 0$$

$$(1 - 16\xi^2)B_4 + 8p_0\xi A_4 + h_0(A_2 - A_6) = 0$$

etc.

Set II:

$$(1 - \xi^2)A_1 - 2p_0\xi B_1 - h_0(B_1 - B_3) = 0$$

$$(1 - \xi^2)B_1 + 2p_0\xi A_1 - h_0(A_1 - A_3) = 0$$

$$(1 - 9\xi^2)A_3 - 6p_0\xi B_3 - h_0(B_1 - B_5) = 0$$

$$(1 - 9\xi^2)B_3 + 6p_0\xi A_3 + h_0(A_1 - A_5) = 0$$

etc.

One solution which would satisfy the equation would be

$A_n = B_n = 0$ for $n = 1, 3, 5, \ldots$ any odd integer.

Such a solution however, would not show the subharmonics A_1 and B_1 which we are seeking. The problem now has become--Find a solution to set II other than

$$A_n = B_n = 0 \quad \text{for odd } n.$$

Assume 1. $w = w_0 \quad \therefore \xi = 1$

2. $|h_0| << 1$

3. $p_0 << 1$

Set I becomes:

(a) $\frac{1}{2}B_0 - h_0A_2 = 0$

(b) $-3A_2 - 4p_0B_2 - h_0(2B_0 - B_4) = G_0/w_0^2$

(c) $-3B_2 + 4p_0A_2 - h_0A_4 = 0$

(d) $15A_4 + 8p_0B_4 + h_0(B_2 - B_6) = 0$

(e) $-15B_4 + 8p_0A_4 + h_0(A_2 - A_6) = 0$

etc.

Set II becomes:

(f) $2p_0B_1 + h_0(B_1 - B_3) = 0$

(g) $2p_0A_1 - h_0(A_1 + A_3) = 0$

(h) $8A_3 + 6p_0B_3 + h_0(B_1 - B_5) = 0$

(i) $-8B_3 + 6p_0A_3 + h_0(A_1 - A_5) = 0.$

The first few terms of the series should give a reasonable approximation.

$\therefore$ Find B_0, A_1, B_1, A_2, B_2

Using (a): $A_2 = \frac{B_0}{2h_0}$

Using (b): $A_2 = -\frac{1}{3}\left[\frac{G_0}{w_0^2} + 4p_0B_2 + 2h_0B_0\right]$ where $h_0B_4 \cong 0$

Using (c): $A_2 = \frac{3B_2}{4P_0}$ where $h_0A_4 \cong 0$

Put (a) and (c) into (b)

$$\frac{3B_2}{4p_0} = -\frac{1}{3}\left\{\frac{G_0}{w_0^2} + 4p_0B_2 + 2\left[2h_0^2\left(\frac{3B_2}{4p_0}\right)\right]\right\}$$

Solving for B_2:

$$\left[\frac{3}{4p_0} + \frac{4p_0}{3} + \frac{3h_0}{p_0}\right] B_2 = -\frac{G_0}{3w_0^2}$$

From previous assumptions

$$\frac{4p_0}{3} + \frac{3h_0^2}{p_0} \cong 0$$

$$\therefore \quad B_2 = -\frac{4p_0G_0}{9w_0^2}$$

But $A_2 = \frac{3B_2}{4p_0}$ From equation (c)

$$\therefore \quad A_2 = -\frac{G_0}{3w_0^2}$$

and $B_0 = 2h_0A_2 = -\frac{2h_0G_0}{3w^2{}_0}$

For a solution other than zero to exist, the determinate of the coefficients of the equations in Set II must vanish.

$$\begin{array}{rcrcrcrcl}
0\,A_1 & + & (2p_0 & h_0)B_1 & + & 0\,A_3 & - & h_0B_3 & = 0 \\
(2p_0 - h_0)A_1 & + & & 0\,B_1 & - & h_0A_3 & + & 0\,B_3 & = 0 \\
0\,A_1 & + & & h_0\,B_1 & + & 8A_3 & + & 6p_0B_3 & = 0 \\
h_0\,A_1 & + & & 0\,B_1 & + & 6p_0A_3 & - & 8B_3 & = 0
\end{array}$$

$$\Delta = \begin{vmatrix} 0 & 2p_0 & h_0 & 0 & -h_0 \\ 2p_0 - h_0 & 0 & -h_0 & 0 & \\ 0 & h_0 & 8 & 6p_0 & \\ h_0 & 0 & 6p_0 & -8 & \end{vmatrix} = 0$$

Expanding:

$$(2p_0 \quad h_0) \begin{vmatrix} 2p_0 - h_0 & -h_0 & 0 \\ 0 & 8 & 6p_0 \\ h_0 & 6p_0 & -8 \end{vmatrix} + h_0 \begin{vmatrix} 2p_0 - h_0 & -h_0 & 0 \\ 0 & 8 & h_0 \\ h_0 & 6p_0 & 0 \end{vmatrix}$$

$$= 0$$

$$(2p_0 \quad h_0)(2p_0 - h_0) \begin{vmatrix} 8 & 6p_0 \\ 6p_0 & -8 \end{vmatrix} + h_0(2p_0 - h_0) \begin{vmatrix} 0 & 6p_0 \\ h_0 & -8 \end{vmatrix}$$

$$- h_0^2 \begin{vmatrix} 2p_0 - h_0 & -h_0 \\ h_0 & 6p_0 \end{vmatrix} = 0$$

thus

$$(4p_0^2 - h_0^2)[-64 - 36p_0^2] + (2p_0h_0 + h_0^2)[-6p_0h_0]$$
$$- h_0^2[12p_0^2 - 6h_0p_0 + h_0^2] = 0$$

arrange this polynomial in powers of h_0

$$- h_0^4 + (6p_0 - 6p_0)h_0^3 + (64 + 36p_0^2 - 12p_0^2 - 12p_0^2)h_0^2$$
$$+ [0]\, h_0 - 256p_0^2 - 144p_0^4 = 0$$

or

$$h_0^4 - (64 + 12p_0^2)h_0^2 + 256p_0^2 + 144p_0^4 = 0$$

let $u = h^2$

$$u^2 - (64 + 12p_0^2)\, u + 256p_0^2 + 144p_0^4 = 0$$

$$U = \frac{\left[64 + 12p_0^2\right] \pm \sqrt{\left[64 + 12p_0^2\right]^2 - 4(256p_0^2 + 144p_0^4)}}{2}$$

$$U = \left[32 + 6p_0^2\right] \pm \left[1024 + 128p_0^2 - 108p_0^4\right]^{\frac{1}{2}}$$

Expand the square root by the binomial theorem.

let $c = 128p_0^2 - 108\,p_0^4$

$$\left[1024 + c\right]^{\frac{1}{2}} = 32 + \tfrac{1}{2}\left(\frac{1}{32}\right)c = 32 + \frac{c}{64}$$

$$= 32 + \frac{128p_0^2 - 108p_0^4}{64}$$

$$= 32 + 2p_0^2 - \frac{27}{16}\,p_0^4$$

$$U = 32 + 6p_0^2 - \left[32 + 2p_0^2 - \frac{27}{16}\,p_0^4\right]$$

$$U = h_0^2$$

$$\therefore \quad h_0^2 = p_0^2\left[4 + \frac{27}{16}\,p_0^2\right]$$

$$\text{or} \quad h_0 = p_0\left[4 + \frac{27}{16}\,p_0^2\right]^{\frac{1}{2}}$$

Expand by the binomial theorem:

$$\pm h_0 = 2p_0 + \frac{27}{64}\,p_0^3$$

$$\text{or} \quad \frac{h_0}{2p_0} = 1 + \frac{27}{128}\,p_0^2$$

by definition

$$\frac{h_0}{2p_0} = \left[\frac{g_1 G_0}{2w_0^2}\right]\left[\frac{w_0}{2}\right] = \left[\frac{g_1 G_0}{2w_0}\right]\frac{1}{2\left[\frac{r_0}{2m_0}\right]}$$

$$= \frac{g_1 F_0}{2w_0 r_0}$$

Using the assumption that $p_0 << 1$

$$\frac{h_0}{2p_0} = \frac{g_1 f_0}{2w_0 r_0} = 1$$

or $$h_0 = 2p_0 = \frac{2p_0 g_1 F_0}{2w_0 r_0}$$

and

$$g_1 F_0 = 2w_0 r_0$$

Evaluating the coefficients for $h_0 = 2p_0$:

It has been previously shown that

$$B_0 = -\frac{2h_0 G_0}{3 w_0^2}$$

$$B_0 = -\frac{2G_0}{3w_0^2}\left[\frac{2p_0 g_1 G_0 m_0}{2w_0 r_0}\right] = -\frac{4G_0^2}{6w_0^3}\,\frac{p_0 g_1}{\frac{r_0}{m_0}}$$

but $$\frac{r_0}{m_0} = 2\gamma = 2p_0 w_0$$

$$B_0 = -\frac{4G_0^2}{6w_0^3}\cdot\frac{g_1}{2w_0} = -\frac{4g_1 G_0^2}{12w_0^4}$$

$$B_0 = -\frac{g_1 G_0^2}{3w_0^4}$$

Using equation (g)

$$2p_0 A_1 - h_0 A_1 = 0 \qquad \text{assuming } A_3 \cong 0$$

$$(2p_0 - h_0) A_1 = 0$$

this leaves A_1 arbitrary.

Using equation (f)

$$2p_0B_1 + h_0B_1 = 0 \qquad B_3 \cong 0$$

$$(2p_0 + h_0)\,B_1 = 0$$

$$\therefore \quad B_1 = 0 .$$

B_2 was previously calculated as :

$$-\frac{4p_0G_0}{9w_0^2}$$

since $p_0 = \frac{\gamma}{w_0}$

$$B_2 = -\frac{4\gamma G_0}{9w_0^3}$$

A_2 was previously calculated as:

$$-\frac{G_0}{3w_0^2}$$

The solution may now be written:

$$x = \frac{B_0}{2} + A_1 \sin wt + A_2 \sin 2wt + B_2 \cos 2wt$$

$$x = -\frac{g_1G_0^2}{6w_0^4} + A_1 \mathrm{Sin}\ wt - \frac{G_0}{3w_0^2} \sin 2wt - \frac{4\gamma G_0}{9w_0^3} \cos 2wt .$$

Consider the equation :

$$m_0 (1 + \mu x) \frac{d^2x}{dt^2} + r_0 (1 + \beta x) \frac{dx}{dt} + s_0(1 + \alpha x)x = F \sin 2wt$$

Or, using previous notation:

$$(1+\mu x)\frac{d^2x}{dt^2} + 2(1+\beta x)\frac{dx}{dt} + w_0^2 (1 + \alpha x)x = G_0 \sin 2wt$$

If you place a further restriction upon μ, β and α such that $\mu = \beta = \alpha$, then a transformation may be made to an equation whose solution is known. This restriction physically says that the coefficient of the linear variation of mass, resistance, and stiffness is the same. It may not be a natural restriction but it looks as though it could be realized in an experimental model.

$$(1 + \mu x)\left[\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + w_0^2 x\right] = G_0 \sin 2wt$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + w_0^2 x = \frac{G_0}{1+\mu x}\sin 2wt$$

but if powers of μ higher than unity are neglected:

$$\frac{1}{1+\mu x} \cong 1 - \mu x$$

let $\mu = -g_1$

then $$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + w_0^2 x = G_0 (1 + g_1 x)\sin 2wt$$

This equation has previously been solved in detail.

Consider the equation:

$$m_0 (1+\mu x)\frac{d^2x}{dt^2} + r_0(1+\beta x)\frac{dx}{dt} + s_0(1+\alpha x)x = F_0(1+g_1 x)\sin 2wt$$

divide through by m_o and make substitutions for $\frac{r_0}{m_0}$, $\frac{s_0}{m_0}$ and $\frac{F_0}{m_0}$

$$(1+\mu x)\frac{d^2x}{dt^2} \quad 2\gamma(1+\beta x)\frac{dx}{dt} + w_0^2(1+\alpha X)X = F_0(1+g_1 x)\sin 2wt$$

To a first approximation this may be written as

$$e^{\mu x}\frac{d^2x}{dt^2} + 2\gamma e^{\beta x}\frac{dx}{dt} + w_0^2 e^{\alpha x}x = G_0(1+g_1x)\sin 2wt$$

If we assume $\mu = \beta = \alpha$

$$e^{\beta x}\left[\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + w_0^2 x\right] = G_0\ (1 + g_1x)\ \sin 2wt$$

or

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + w_0^2x = G_0e^{-\beta x}\ (1 + g_1x)\sin 2\ wt$$

let $\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + w_0^2\right]x = \square x$

Expand $e^{-\beta x}$ and neglect terms of 2nd order

$$\square x = G_0\ (1 - \beta x)\ \ (1+ g_1x)\ \sin 2wt.$$

$$= G_0\left[1 + (g_1 - \beta)x\right]\ \sin 2wt.$$

let $g_1 - \beta = a$

$$\square x = G_0\left[1 + ax\right]\ \sin 2wt$$

This problem has been solved previously.

The generalized equation is:

$$m_0\ (1+\mu x)\frac{d^2x}{dt^2} + r_0(1+\beta x)\frac{dx}{dt} + s_0(1+\alpha x) = F_0(1+g_1x)\sin 2wt$$

divide through by m_0 and use the same notation employed in the previous case.

$$(1+\mu x)\frac{d^2x}{dt^2}\ \ 2\gamma\ (1+\beta x)\frac{dx}{dt} + w_0^2(1+\alpha x)x = G_0(1+g_1x)\sin 2wt$$

Assume small amplitudes

Try as a solution

$$x = \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \sin nwt + B_n\cos nwt$$

Substitute the $\sum$ into the differential equation.

$$\left[1+\mu\left(\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right)\right]\left[-\sum_{n=1}^{\infty} n^2w^2(A_n \sin nwt + B_n \cos nwt)\right]$$

$$+2\gamma\left[1+\beta\left(\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right)\right]\left[\sum_{n=1}^{\infty} nw(A_n \cos nwt - B_n \sin nwt)\right]$$

$$+w_0^2\left[1+\alpha\left(\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right)\right]\left[\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right]$$

$$=G_0\left[1+g_1\left(\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right)\right] \sin 2wt$$

Treating the left side only:

$$\left[1+\mu\frac{B_0}{2}\right]\left[\sum_{n=1}^{\infty} -n^2w^2(A_n \sin nwt + B_n \cos nwt)\right]$$

$$-\mu\left[\sum_{n=1}^{\infty} n^2w^2(A_n \sin nwt + B_n \cos nwt)\right]\left[\sum_{k=1}^{\infty} A_k \sin kwt + B_k \cos kwt\right]$$

$$+2\gamma\left[1+\beta\frac{B_0}{2}\right]\left[\sum_{n=1}^{\infty} nw(A_n \cos nwt - B_n \sin nwt)\right]$$

$$+2\gamma\beta\left[\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right]\left[\sum_{k=1}^{\infty} kw(A_k \cos kwt - B_k \sin kwt)\right]$$

$$+w_0^2\left[1+\alpha\frac{B_0}{2}\right]\left[\frac{B_0}{2}+\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right]$$

$$+w_0^2\alpha\left[\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right]\left[\frac{B_0}{2}+\sum_{k=1}^{\infty} A_k \sin kwt + B_k \cos kwt\right]$$

Treating only the various products of two series in the above expression.

$$-\mu\left[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (n^2w^2A_n \sin nwt)(A_k \sin kwt) + (n^2w^2A_n \sin nwt)\right.$$

$(B_k \cos kwt) + (n^2w^2B_n \cos nwt)(A_k \sin kwt) + (n^2w^2B_n \cos nwt)$

$(B_k \cos kwt)\Big]$

Each term maybe considered as a separate $\sum$ thus n and k may be interchanged in any term.

$$-\mu\left[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} n^2w^2A_nA_k \sin nwt \sin kwt + (n^2 + k^2)w^2A_nB_k \sin nwt \cos kwt + n^2w^2B_nB_k \cos nwt \cos kwt\right]$$

This is the complete expression for the product of the two series in the μ term.

2 γ term: Working only with the products of two series.

$$2\gamma\beta\left[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} (A_n \sin nwt)(kw\,A_k \cos kwt) - (A_n \sin nwt)(kwB_k \sin kwt) + (B_n \cos nwt)(kw\,A_k \cos kwt) - (B_n \cos nwt)(kw\,B_k \sin kwt)\right]$$

or

$$2\gamma\beta\left[\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} kwA_kB_n \cos kwt \cos nwt + (k\,A_nA_k - nB_nB_k)w \sin nwt \cos kwt - kw\,A_nB_k \sin nwt \sin kwt\right]$$

This is the complete expression for the product of the two series in the 2 γ term.

w_o^2 term: Consider only the product of series.

$$w_o^2\alpha\left[\sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt\right]\left[\sum_{k=1}^{\infty} A_k \sin kwt + B_k \cos kwt\right]$$

$$w_0^2\alpha\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} (A_n \sin nwt)(A_k \sin kwt) + (A_n \sin nwt)(B_k \cos kwt) + (B_n \cos nwt)(A_k \sin kwt) + (B_n \cos nwt)(B_k \cos kwt)\right]$$

or

$$W_0^2\alpha\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_nA_k \sin nwt \sin kwt + 2A_nB_k \sin nwt \cos kwt + B_nB_k \cos nwt \cos kwt\right]$$

This is the expression for the product of the two series in the w_0^2 terms.

Collect terms in the $\Sigma\Sigma$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[-\mu n^2w^2A_nA_k - 2\gamma\beta kwA_nB_k + w_0^2\alpha A_nA_k\right]$$

$$\sin nwt \sin kwt + \left[-\mu(n^2+k^2)w^2A_nB_k + 2\gamma\beta(kA_nA_k - nB_nB_k)w 2w_0^2\alpha A_nB_k\right] \sin nwt \cos kwt + \left[-\mu n^2w^2B_nB_k + 2\gamma\beta kwA_kB_n + w_0^2\alpha B_nB_k\right] \cos nwt \cos kwt.$$

or

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[(w_0^2\alpha - \mu n^2w^2)A_nA_k - 2\gamma\beta kwA_nB_k\right] \sin nwt \sin kwt + \left\{\left[2w_0^2\alpha - \mu(n^2+k^2)w^2\right]A_nB_k + 2\gamma\beta(kA_nA_k - nB_nB_k)w\right\} \sin nwt \cos kwt + \left[(w_0^2\alpha - \mu n^2w^2)B_nB_k + 2\gamma\beta kwA_kB_n\right] \cos nwt \cos kwt.$$

From trigonometry

$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[(w_0^2\alpha-\mu n^2w^2)A_nA_k-2\gamma\beta kwA_nB_k\right]\Big[\cos(n-k)wt$$

$$-\cos(n+k)wt+\Big\{\left[2w_0^2\alpha-\mu(n^2+k^2)w^2\right]A_nB_k$$

$$+2\gamma\beta\left[kA_nA_k-nB_nB_k\right]w\Big\}\left[\sin(k+n)wt+\sin(n-k)wt\right]$$

$$+\left[(w_0^2\alpha-\mu n^2w^2)B_nB_k+2\gamma\beta kwA_kB_n\right]\left[\cos(n-k)wt+\cos(n+k)wt\right]$$

Factoring

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big[(w_0^2\alpha-\mu n^2w^2)A_nA_k-2\gamma\beta kwA_nB_k+(w_0^2\alpha-\mu n^2w^2)B_nB_k$$

$$+2\gamma\beta kwA_kB_n\Big]\cos(n-k)wt+\Big[(w_0^2\alpha-\mu n^2w^2)B_nB_k+2\gamma\beta kwA_kB_n$$

$$-\left[w_0^2\alpha-\mu n^2w^2)A_nA_k-2\gamma\beta kwA_nB_k\right]\cos(n+k)wt$$

$$+\Big[\left[2w_0^2\alpha-\mu(n^2+k^2)w^2\right]A_nB_k+2\gamma\beta w(kA_nA_k-nB_nB_k)\Big]\Big[\sin(n+k)wt$$

$$+\sin(n-k)wt\Big]$$

or

$$\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big[(w_0^2\alpha-\mu n^2w^2)(A_nA_k+B_nB_k)+2\gamma\beta kw(A_kB_n-A_nB_k)\Big]$$

$$\cos(n-k)wt+\Big[(w_0^2\alpha-\mu n^2w^2)(B_nB_k-A_nA_k)+2\gamma\beta kw(A_kB_n+A_nB_k)\Big]$$

$$\cos(n+k)wt+\Big\{\left[2w_0^2\alpha-\mu(n^2+k^2)w^2\right]A_nB_k+2\gamma\beta w(kA_nA_k-nB_nB_k)\Big\}$$

$$\left[\sin(n+k)wt+\sin(n-k)wt\right]$$

The complete substitution may now be written:

$$w_0^2\left(1+\frac{\alpha B_0}{2}\right)\frac{B_0}{2}+\sum_{n=1}^{\infty}\Big\{\Big[-\left(1+\mu\frac{B_0}{2}\right)n^2w^2A_n-2\gamma\left(1+\beta\frac{B_0}{2}\right)nwB_n$$

$$+w_0^2\left(1+\frac{\alpha B_0}{2}\right)A_n+w_0^2\left[\alpha\frac{B_0}{2}\right]A_n\Big]\sin nwt+\Big[-\left(1+\mu\frac{B_0}{2}\right)n^2w^2B_n$$

$$+2\gamma\left(1+\frac{\beta B_0}{2}\right)nwA_n + w_0^2\left(1+\frac{\alpha B_0}{2}\right)B_n + w_0^2\frac{\alpha B_0}{2}B_n\Big]\cos nwt$$

$$+\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big[(w_0^2\alpha-\mu n^2w^2)(A_nA_k+B_nB_k)+2\gamma\beta kw(A_kB_n-A_nB_k)\Big]$$

$$\cos(n-k)\,wt+\Big[(w_0^2\alpha-\mu n^2w^2)(B_nB_k-A_nA_k)+2\gamma\beta kw(A_kB_n+A_nB_k)\Big]$$

$$\cos(n+k)\,wt+\Big\{\Big[2w_0^2\alpha-\mu(n^2+k^2)w^2\Big]A_nB_k$$

$$+2\gamma\beta w(kA_nA_k-nB_nB_k)\Big\}\Big[\sin(n+k)wt+\sin(n-k)wt\Big]=$$

$$\left[G_0+\frac{G_0g_1B_0}{2}\right]\sin 2wt+\frac{G_0g_1}{2}\sum_{n=1}^{\infty}A_n\Big[\cos(n-2)wt-\cos(n+2)wt$$

$$B_n\Big[\sin(n+2)wt-\sin(n-2)wt\Big]$$

Factoring:

$$\frac{B_0w_0^2}{2}\left(1+\frac{\alpha B_0}{2}\right)+\sum_{n=1}^{\infty}\Big\{A_n\Big[w_0^2(1+\alpha B_0)-n^2w^2\left(1+\frac{\mu B_0}{2}\right)\Big]$$

$$-2\gamma\left(1+\frac{\beta B_0}{2}\right)nwB_n\Big\}\sin nwt+\Big\{\Big[w_0^2(1+\alpha B_0)-n^2w^2\left(1+\frac{\mu B_0}{2}\right)\Big]B_n$$

$$+2\gamma\left(1+\frac{\beta B_0}{2}\right)nwA_n\Big\}\cos nwt+\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big[(w^2\alpha-\mu n^2w^2)$$

$$(A_nA_k+B_nB_k)+2\gamma\beta kw(A_kB_n-A_nB_k)\Big]\cos(n-k)\,wt$$

$$+\Big[(w_0^2\alpha-\mu n^2w^2)(B_nB_k-A_nA_k)+2\gamma\beta w(A_kB_n+A_nB_k)\Big]$$

$$\cos(n+k)\,wt+\Big\{\Big[2w_0^2\alpha-\mu(n^2+k^2)w^2\Big]A_nB_k$$

$$+2\gamma\beta w(kA_nA_k-nB_nB_k)\Big\}\Big[\sin(n+k)wt+\sin(n-k)wt\Big]=$$

$$\left[G_0+\frac{G_0g_1B_0}{2}\right]\sin 2wt+\frac{G_0g_1}{2}\sum_{n=1}^{\infty}A_n\Big[\cos(n-2)wt-$$

$$\cos(n+2)wt\Big]+B_n\Big[\sin(n+2)wt-\sin(n-2)wt\Big]$$

SPECIAL CASES

I

$\beta = \alpha = g_1 = 0$

The general expression reduces to:

$$\frac{B_0 w_0^2}{2} + \sum_{n=1}^{\infty} \left\{ A_n \left[w_0^2 - n^2 w^2 \left(1 + \mu \frac{B_0}{2} \right) \right] - 2\gamma n w B_n \right\} \sin nwt$$

$$+ \left\{ \left[w_0^2 - n^2 w^2 \left(1 + \mu \frac{B_0}{2} \right) \right] B_n + 2\gamma n w A_n \right\} \cos nwt$$

$$+ \tfrac{1}{2} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[-\mu n^2 w^2 (A_n A_k + B_n B_k) \right] \cos (n-k)wt$$

$$+ \left[-\mu (n^2 + k^2) w^2 A_n B_k \right] \left[\sin(n+k)wt + \sin(n-k)wt \right]$$

$$+ \left[-\mu n^2 w^2 (B_n B_k - A_n A_k) \right] \cos(n+k)wt = G_0 \sin 2wt$$

Constant term:

$$\frac{B_0 w_0^2}{2} + \tfrac{1}{2} \sum_{n=1}^{\infty} -\mu n^2 w^2 (A_n^2 + B_n^2) + \tfrac{1}{2} \sum_{k=1}^{\infty} -\mu k^2 w^2 (A_k^2 + B_k^2) = 0$$

or

$$\frac{B_0 w_0^2}{2} - \sum_{n=1}^{\infty} \mu n^2 w^2 (A_n^2 + B_n^2) = 0$$

Coefficient of sin wt:

$$A_1 \left[w_0^2 - w^2 \left(1 + \mu \frac{B_0}{2} \right) \right] - 2\gamma w B_1 - \mu \sum_{n=1}^{\infty} \left[n^2 + (n-1)^2 \right] w^2 A_n B_{n-1}$$

$$+ \mu \sum_{k=1}^{\infty} \left[k^2 + (k-1)^2 \right] w^2 A_{k-1} B_k = 0$$

$$A_1 (w_0^2 - w^2) - 2\gamma w B_1 + \mu \left[-\frac{A_1 B_0}{2} + \sum_{n=1} \left[n^2 + (n-1)^2 \right] \left[A_{n-1} B_n - A_n B_{n-1} \right] \right]$$

$$= 0$$

Coefficient of cos wt:

$$B_1(w_0^2 - w^2) + 2\gamma w A_1 + \mu\left[-\frac{B_0 B_1}{2} - \sum_{n=1}^{\infty} \mu n^2 w^2 (A_n A_{n-1} + B_n B_{n-1})\right] = 0$$

Coefficient of sin 2wt:

$$A_2\left[w_0^2 - 4w^2\left(1 + \frac{\mu B_0}{2}\right)\right] - 4\gamma w B_2 - \frac{2}{2}\mu w^2 A_1 B_1$$
$$- \sum_{n=1} \left[n^2 + (n-2)^2\right] w^2 \left[A_n B_{n-2} - A_{n-2} B_n\right] = G_0$$

or

$$A_2\left[w_0^2 - 4w^2\right] - 4\gamma w B_2 - w^2\mu\left[2A_2 B_0 + A_1 B_1\right.$$
$$\left. + \sum_{n=1}^{\infty} \left[n^2 + (n-2)^2\right]\left[A_n B_{n-2} - A_{n-2} B_n\right]\right] = G_0$$

Coefficient of cos 2wt:

$$B_2\left[w_0^2 - 4w^2\left(1 + \mu\frac{B_0}{2}\right)\right] + 4\gamma w A_2 - \mu w^2 (B_1^2 - A_1^2)$$
$$+ \sum_{n=1}^{\infty} - \mu n^2 w^2 \left[A_n A_{n-2} + B_n B_{n-2}\right] = 0$$

or

$$B_2\left[w_0^2 - 4w^2\right] + 4\gamma w A_2 - \mu w^2\left[-A_1^2 + B_1^2 + \sum_{n=1}^{\infty} n^2 (A_n A_{n-2} + B_n B_{n-2})\right] = 0$$

II

$\mu = \beta = g_1 = 0$

The general expression becomes:

$$\frac{B_0 w_0^2}{2}\left[1 + \frac{\alpha B_0}{2}\right] + \sum_{n=1}^{\infty}\left\{A_n\left[w_0^2(1+\alpha B_0) - n^2w^2\right] - 2\gamma nwB_n\right\}\sin nwt + \left\{\left[w_0^2(1+\alpha B_0) - n^2w^2\right]B_n + 2\gamma nwA_n\right\}\cos nwt + \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[w_0^2\alpha(A_nA_k + B_nB_k)\right]$$

$$\cos(n-k)wt + \left[w_0^2\alpha(B_nB_k - A_nA_k)\right]\cos(n+k)wt$$

$$2w_0^2\alpha A_nB_k\left[\sin(n+k)wt + \sin(n-k)wt\right] = G_0 \sin 2wt$$

Constant term:

$$\frac{B_0 w_0}{2}\left[1 + \frac{\alpha B_0}{2}\right] + \tfrac{1}{2}\sum_{n=1}^{\infty} w_0^2\alpha(A_nA_{n-1} + B_nB_{n-1}) + \tfrac{1}{2}\sum_{k=1}^{\infty} w_0^2\alpha(A_{k-1}A_k + B_{k-1}B_k) = 0$$

or

$$\frac{B_0 w_0^2}{2}\left[1 + \frac{\alpha B_0}{2}\right] + \sum_{n=1}^{\infty} w_0^2\alpha(A_nA_{n-1} + B_nB_{n-1}) = 0$$

Coeffident of sin wt:

$$A_1\left[w_0^2(1+\alpha B_0) - w^2\right] - 2\gamma wB_1 + \tfrac{1}{2}\sum_{n=1}^{\infty} 2w_0^2\alpha A_nB_{n-1} - \tfrac{1}{2}\sum_{k=1}^{\infty} 2w_0^2\alpha A_{k-1}B_k = 0$$

Coefficient of the cos wt:

$$B_1\left[w_0^2 (1 + \alpha B_0) - w^2\right] + 2\gamma w A_1 + \tfrac{1}{2} \sum_{n=1}^{\infty} w_0^2 \alpha (A_n A_{n-1} + B_n B_{n-1})$$

$$+ \tfrac{1}{2} \sum_{k=1}^{\infty} w_0^2 \alpha (A_{k-1} A_k + B_{k-1} B_k) = 0$$

or

$$B_1\left[w_0^2 (1 + \alpha B_0) - w^2\right] + 2\gamma w A_1 + \sum_{n=1} w_0^2 \alpha (A_n A_{n-1} + B_n B_{n-1}) = 0$$

Coefficient of the sin 2wt:

$$A_2\left[w_0^2 (1 + \alpha B_0) - 4w^2 - 4\gamma w B_2 + \frac{2w_0 \alpha A_1 B_1}{2}\right.$$

$$+ \tfrac{1}{2} \sum_{n=1}^{\infty} 2w_0^2 \alpha A_n B_{n-2} - \tfrac{1}{2} \sum_{k=1}^{\infty} 2w_0^2 \alpha A_{k-2} B_k = G_0$$

Coefficient of the cos 2 wt:

$$B_2\left[w_0^2 (1 + \alpha B_0) - 4w^2\right] + 4\gamma w A_2 + \frac{w_0^2 \alpha (B_1^2 - A_1^2)}{2}$$

$$+ \tfrac{1}{2} \sum_{n=1}^{\infty} w_0^2 \alpha (A_n A_{n-2} + B_n B_{n-2}) + \tfrac{1}{2} \sum_{k=1}^{\infty} w_0^2 \alpha (A_{k-2} A_k + B_{k-2} B_k) = 0$$

III

$\mu = \alpha = g_1 = 0$

The general expression reduces to:

$$\frac{B_0 w_0^2}{2} + \sum_{n=1}^{\infty}\left\{A_n\left[w_0^2 - n^2w^2\right] + 2\gamma\left[1 + \frac{\beta B_0}{2}\right] nwB_n\right\}\sin nwt$$

$$+ \left\{\left[w_0^2 - n^2w^2\right] B_n + 2\gamma\left[1 + \frac{\beta B_0}{2}\right] nwA_n\right\}\cos nwt$$

$$+ \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} 2\gamma\beta kw(A_kB_n - A_nB_k)\cos(n-k)wt$$

$$+ 2\gamma\beta kw\,(A_kB_n + A_nB_k)\cos(n+k)wt$$

$$+ 2\gamma\beta w(kA_nA_k - nB_nB_k)\left[\sin(n+k)wt + \sin(n-k)wt\right] =$$

$G_0 \sin 2wt.$

Constant term:

$$\frac{B_0 w_0^2}{2} + \sum_{n=1}^{\infty} 2\gamma\beta nw(A_nB_n - A_nB_n) = 0$$

$$B_0 w_0^2 = 0$$

Coefficient of sin wt:

$$A_1\left[w_0^2 - w^2\right] - 2w\gamma B_1 - \gamma\beta wB_0B_1$$

$$+ \sum_{n=1}^{\infty}\gamma\beta w\left[(n-1)A_nA_{n-1} - nB_nB_{n-1}\right] - \sum_{k=1}^{\infty}\gamma\beta w\left[kA_{k-1}A_k -\right.$$

$$\left. kA_{k-1}A_k - (k-1)B_kB_{k-1}\right] = 0$$

Coefficient of cos wt:

$$B_1(w_0^2 - w^2) + 2\gamma wA_1 + \gamma w\beta B_0B_1 + \sum_{n=1}^{\infty}\gamma\beta w(n-1)\left[B_nA_{n-1} - A_nB_{n-1}\right]$$

$$+ \sum_{k=1}^{\infty}\gamma\beta w\left[A_kB_{k-1} - B_kA_{k-1}\right] = 0$$

Coefficient of sin 2wt:

$$A_2\left[(w^2 - 4w^2)\right] - 4\gamma w B_2 - 2\gamma\omega\beta B_0 B_2 + 2\gamma w\beta\left[A_1^2 - B_1^2\right]$$
$$+ \sum_{n=1}^{\infty}\gamma\beta w\left[(n-1)A_n A_{n-1} - nB_n B_{n-1}\right] - \sum_{k=1}^{\infty}\gamma w\beta\left[kA_k A_{k-1} - (k-1)B_k B_{k-1}\right]$$

Coefficient of cos 2wt:

$$B_2\ (w_0^2 - 4w^2) + 4\gamma w A_2 + 2\gamma w B_0 A_2$$
$$+ \tfrac{1}{2}\left[2\gamma\beta w\ (2A_1 B_1) + \sum_{n=1}^{\infty}\gamma\beta w(n-2)\ (A_{n-2}B_n - A_n B_{n-2})\right.$$
$$+ \tfrac{1}{2}\sum_{k=1}^{\infty}\gamma\beta k w\ (A_k B_{k-2} - A_{k-2}B_k)$$

The solution for these special cases may be obtained by using a procedure similar to the problem previously considered.

Outline of an Iterative Method (3)

$$\frac{d^2x}{dt^2} + w^2x = \mu f(x, \dot{x}) + a \sin nwt$$

Assume a solution of the form

$$x = x^{(0)} + \mu x^{(1)} + \mu^2 x^{(2)} + \cdots$$

$x^{(0)}$ is obtained by assuming $\mu = 0$

The equation becomes:

$$\frac{d^2x^{(0)}}{dt^2} + w^2x^{(0)} = a \sin nwt$$

The solution is:

$$x^{(0)} = c^{(0)} \sin (wt - \alpha^{(0)}) + \frac{a}{w^2(1-n^2)} \sin nwt$$

$x^{(1)}$ is obtained by substituting $x^{(0)}$ into the right side of the equation and equating coefficients of like powers of μ.

$$\mu\left[\frac{d^2x^{(1)}}{dt^2} + w^2x^{(1)}\right] = \mu f(x^{(0)}, \dot{x}^{(0)}) + a \sin nwt$$

or

$$\frac{d^2x^{(1)}}{dt^2} + w^2x^{(1)} = f(x^{(0)}, \dot{x}^{(0)})$$

Initial conditions at $t = 0$ $\begin{cases} x^{(1)} = 0 \\ \frac{dx^{(1)}}{dt} = 0 \end{cases}$

The solution of this equation which satisfies the initial condition is:

$$x^{(1)} = \int_0^t f\left[x^{(0)}(\tau), \dot{x}^{(0)}(\tau)\right] \cdot \sin w(t - \tau)\, d\tau$$

This solution corresponds to a periodic motion of period $\frac{2\pi}{w}$ if:

$$x^{(1)}\left(\frac{2\pi}{w}\right) = -\int_0^{2\pi} f\left[x^{(0)}(\tau),\ \dot{x}^{(0)}(\tau)\right]\cdot \sin w\tau\, d\tau = 0$$

and

$$\dot{x}^{(1)}\left(\frac{2\pi}{w}\right) = \int_0^{2\pi} f\left[x^{(0)}(\tau),\ \dot{x}^{(0)}(\tau)\right]\cdot \cos w\tau\, d\tau = 0$$

This set of three equations may be solved, when $f(x,\dot{x})$ is given, for $c^{(0)}$, $\alpha^{(0)}$, and $x^{(1)}$.

Since $\dot{x}^{(0)}$ and $x^{(0)}$ are periodic functions it is possible to expand $f(x^{(0)},\ \dot{x}^{(0)})$ into a Fourier series of the form,

$$x^{(1)} = c^{(1)}\sin(wt - \alpha^{(1)}) + \sum_{n=2}^{\infty} \frac{a_n \sin nwt}{w^2(1-n^2)} + \frac{b_0}{w^2}$$

$$+ \sum_{n=2}^{\infty} \frac{b_n \cos nwt}{w^2(1-n^2)}$$

Thus it has been shown that subharmonic resonance will be present in a large class of vibrating systems.

Solution by the Method of Differences:

$$\frac{d^2x}{dt^2} + w_o^2 x = f(x, \dot{x}, t)$$

By the use of numerical methods values of x may be found for given values of t. Using difference methods derivatives of x may be calculated to any order.

Using the above information a Taylor Series may be written for f(x).

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

let $h = x-a$

$$f(x) = \sum_{n=0}^{\infty} \frac{h^n f^{(n)}(a)}{n!}$$

The calculated values should be taken as close to the portion of the curve being considered as possible.

Many times error is introduced when calculating derivatives by the method of differences so that care should be taken to check on this error.

Since a periodic solution is desired we may write:

$$f(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \sin nwt + B \cos nwt$$

The coefficients of such a series would be given by the ordinary methods.

If $$f(x) = \sum_{n=0}^{\infty} \frac{h^n f^{(n)}(a)}{n!}$$

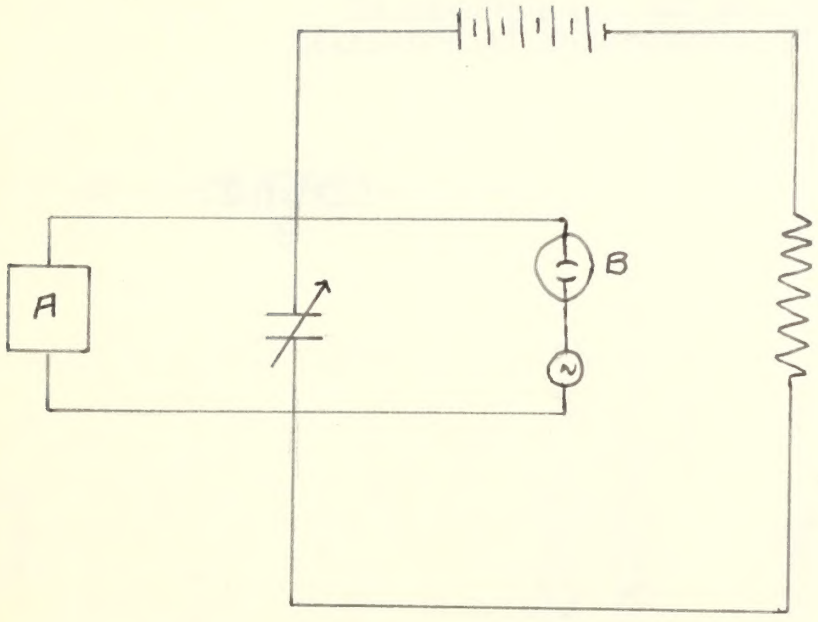
then

$$\int_0^T \left[\sum_{m=0}^{\infty} \frac{h^m f^{(m)}(a)}{m!} \right] \sin n\omega t \, dt = A_n \int_0^T \sin^2 n\omega t \, dt$$

$$\int_0^T \left[\sum_{m=0}^{\infty} \frac{h^m f^{(m)}(a)}{m!} \right] \cos n\omega t \, dt = B_n \int_0^T \cos^2 n\omega t \, dt$$

$$\frac{1}{T} \int_0^T \left[\sum_{m=0}^{\infty} \frac{h^m f^{(m)}(a)}{m!} \right] dt = \frac{B_0}{2}$$

EXPERIMENT I



A - Oscilloscope

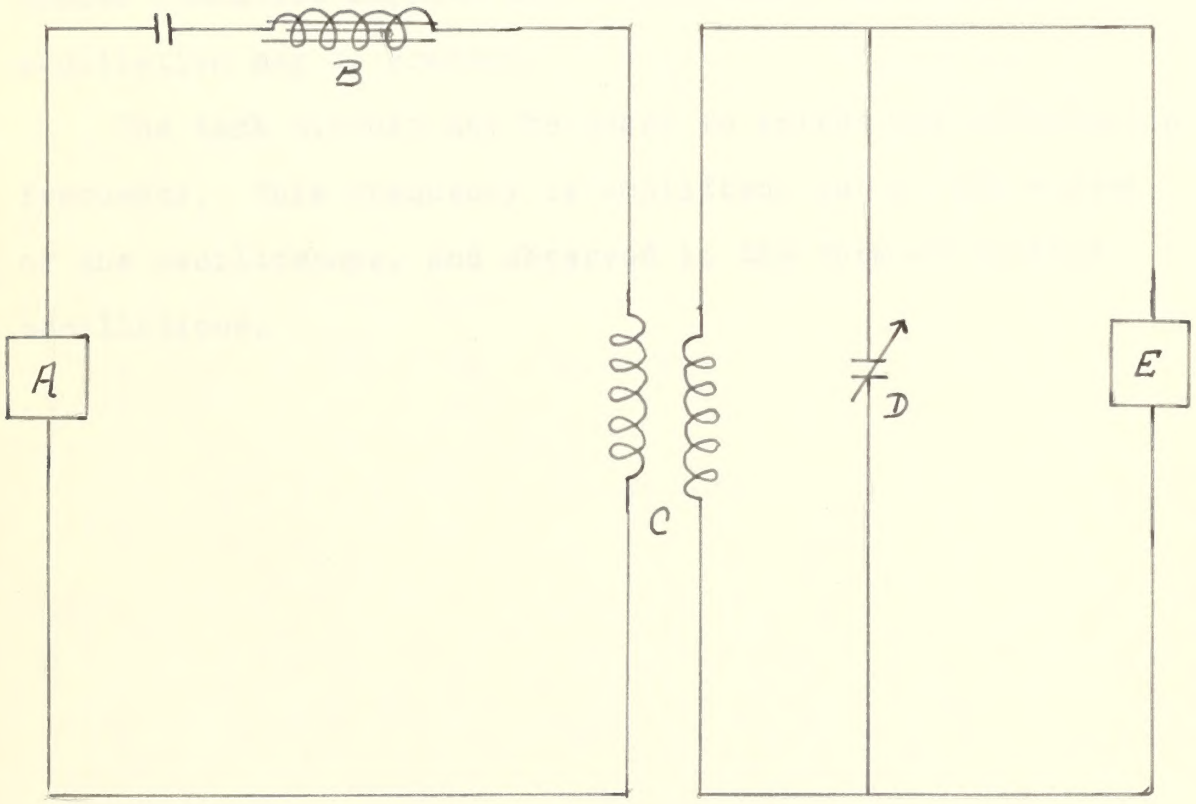
B - Neon Bulb

This experiment is described in Nature Magazine of September 10, 1927. It was claimed that subharmonics up to 1/40 of the driving frequency were observed. When this was set up it yielded no subharmonics of any sort.

R
G

EXPERIMENT II

To Demonstrate the Actual
Existence of Subharmonic Resonance



A - High frequency generator

B - Iron cored inductance

6 - Air cored transformer

D - Variable parallel resonant circuit

E - Amplifier and Oscilloscope

This experiment is designed to generate and detect subharmonic frequencies.

The varying permeability of the iron cored inductance produces non-linearity in the primary circuit. This non-linear condition implies that subharmonic frequencies of oscillation may be present.

The tank circuit may be tuned to select any subharmonic frequency. This frequency is amplified, put on the screen of the oscilloscope, and observed in the form of visible oscillations.

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Mathematical Analysis of the Experiments

Reference Figure Experiment I:

Assume that the voltage drop across the neon bulb is a function of q.

Kirchhoff's law gives the following:

$$E \sin nwt - f(q) - \frac{q}{c} = 0$$

$$\frac{q}{c} = -f(q) + E \sin nwt$$

let $F(q) = -f(q)$

differentiating this equation gives

$$\frac{1}{c}\frac{dq}{dt} = F^{(1)}(q)\frac{dq}{dt} + nw\,E\cos nwt$$

or

$$\left[\frac{1}{c} - F^{(1)}(q)\right]\frac{dq}{dt} = nw\,E\cos nwt$$

since $\frac{dq}{dt} = i$

$$i = \frac{nwE}{c}\cos nwt\left[\frac{1}{1-c\,F^{(1)}(q)}\right]$$

$$i = \frac{nwE}{c}\cos nwt\left[1 + cF^{(1)}(q) + c^2F^{(2)}(q) + \ldots\right]$$

assume $c << 1$

Then the series may be shown to converge and is a function of q. Designate this function as Q(q). It can be expanded into a Fourier series.

$$Q(q) = B_0 + \sum_{n=1}^{\infty} A_n \sin nwt + B_n \cos nwt$$

This series is multiplied by cos nwt.

It is known that

$$\cos nwt \cdot \cos(n-1)wt = \tfrac{1}{2}\cos wt + \tfrac{1}{2}\cos(2n-1)wt$$

This shows that subharmonic terms will be present.

Variation of Experiment I:

Place an inductance in series with the condenser.

The equation describing the circuit becomes:

$$E \sin nwt - f(q,\dot{q}) - \frac{q}{c} - L\frac{di}{dt} = 0$$

Here the function describing the voltage drop across the neon bulb may be made more general: $f(q,\dot{q})$,

or $$L\frac{di}{dt} + \frac{q}{c} = -f(q,\dot{q}) + E \sin nwt$$

let $$F(q,\dot{q}) = -f(q,\dot{q})$$

$$L\frac{di}{dt} + \frac{q}{c} = F(q,\dot{q}) + E \sin nwt$$

but $$\frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$\therefore \quad \frac{d^2q}{dt^2} + \frac{q}{Lc} = \frac{1}{L}F(q,\dot{q}) + \frac{E}{L}\sin nwt$$

This is exactly the same form that was considered under the iterative procedure. Here again theory predicts that subharmonic resonance will be observed.

Reference Figure Experiment II:

Assume that the effect of the secondary circuit on the primary circuit is negligible.

In general for an iron core inductance $i = f(\phi)$

In this case assume

$$i = \frac{\phi}{L} + \frac{1}{b}\phi^3$$

$N_1 + N_2$ is a constant N

$$E \sin nwt = N\frac{d\phi}{dt} + \frac{q}{c} + iR$$

$$nwE \cos nwt = N\frac{d^2\phi}{dt^2} + \frac{1}{c}\frac{dq}{dt} + R\frac{di}{dt}$$

let $\frac{nwE}{N} = A$; $\frac{1}{Nc} = B$; $\frac{R}{N} = D$

$$A \cos nwt = \frac{d^2\phi}{dt^2} + Bi + D\frac{di}{dt}$$

Substituting for i gives:

$$\frac{d^2\phi}{dt^2} + D\left[\frac{1}{L} + \frac{3\phi^2}{b}\right]\frac{d\phi}{dt} + B\left[\frac{\phi}{L} + \frac{\phi^3}{b}\right] = A \cos nwt$$

This equation may be arranged in the following manner:

$$\frac{d^2\phi}{dt^2} + \frac{B}{L}\phi = -D\left[\frac{1}{L} + \frac{3\phi^2}{b}\right]\frac{d\phi}{dt} - \frac{B\phi^3}{b} + A \cos nwt$$

This is the same form as was considered in the iterative method. This method showed that subharmonic resonance would be present.

APPENDIX

A Short Introduction to the LaPlace Transform and its Properties.

The LaPlace transform has found much favor in recent years as a powerful tool in the solution of differential equations. The uniqueness of this tool displays itself in a twofold manner. In the first place, only relatively simple integration is required to solve a differential equation by this method. This is often a great saving of time and effort. The second advantage is seen in the fact that the initial conditions are used directly in the solution of the differential equation rather than considered seperately as is the case in conventional methods of solution.

DEFINITION: LaPlace transform of f(t) is $\int_0^\infty e^{-pt} f(t)dt$.

Thus the transform is a function of p over the range of p for which the transform exists.

There are many conditions to be placed on the function f(t) to ensure the existence of the integral, but all of these conditions are satisfied by the physical phenomena treated in this paper. Thus in all the cases considered here the integral will be defined over the positive range of p.

Properties that will be used in this paper:

Let $L\{t\}$ = LaPlace transform of f(t).

1. $L\{af(t) + bg(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$

2. $L\left\{\frac{d^n f(t)}{dt}\right\} = p^n L\{f(t)\} - p^{n-1} f(0) - p^{n-2} \frac{d f(0)}{dt} - \ldots - \frac{d^{n-1} f(0)}{dt^{n-1}}$.

Examples of LaPlace transforms:

$$L\{1\} = \int_0^\infty 1\, e^{-pt} dt = \frac{-e^{-pt}}{p}\Big|_0^\infty = \frac{1}{p} \qquad p > 0$$

$$L\{e^{at}\} = \int_0^\infty e^{at}\, e^{-pt}\, dt = \int_0^\infty e^{-(p-a)}\, dt = \frac{1}{p-a} \qquad p > a$$

Example of a solution to a simple differential equation with given initial conditions:

$$\frac{d^2y}{dt^2} = a^2 y \qquad \text{where "a" is a constant}$$

$\therefore$

$$\frac{d^2y}{dt^2} - a^2 y = 0$$

Initial Conditions: at $t = 0$ $\begin{cases} y = 1 \\ \frac{dy}{dt} = 0 \end{cases}$

thus,

$$\int_0^\infty e^{-pt} \left[\frac{d^2y}{dt^2} - a^2 y\right] dt = 0$$

or

$$\int_0^\infty e^{-pt} \frac{d^2y}{dt^2}\, dt - a^2 \int_0^\infty e^{-pt}\, y\, dt = 0$$

Integrating the first integral by parts twice

$$\int_0^\infty e^{-pt} \frac{d^2y}{dt^2}\, dt = e^{-pt} \frac{dy}{dt}\Big|_0^\infty + p\, e^{-pt} y\Big|_0^\infty + p^2 \int_0^\infty e^{-pt}\, y\, dt,$$

notice that:

1. $e^{-pt} \frac{dy}{dt}\Big|_0^\infty = -\frac{dy}{dt}(0) = 0$ due to the initial conditions.

2. $pe^{-pt} y\Big|_0^\infty = -py(0) = -p$

Examples of Laplace Transforms:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-pt}dt = \left[\frac{-e^{-pt}}{p}\right]_0^\infty = \frac{1}{p} \qquad p > 0$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}\, e^{-pt}\, dt = \int_0^\infty e^{-(p-a)}\, dt = \frac{1}{p-a} \qquad p > a$$

Example of a solution to a simple differential equation with given initial conditions:

$$\frac{dS}{dt}\lambda = a_S\,\lambda \qquad \text{where "a" is a constant}$$

...

$$\frac{dS}{dt}\lambda - a_S\,\lambda = 0$$

Initial Conditions: at $t = 0$ $\begin{cases} \lambda = 1 \\ \dfrac{d\lambda}{dt} = 0 \end{cases}$

Thus,

$$\int_0^\infty e^{-pt}\left[\frac{d}{dt}\frac{S}{dt}\lambda - a_S\,\lambda\right] dt = 0$$

or

$$\int_0^\infty e^{-pt}\frac{d}{dt}\frac{S}{dt}\lambda\, dt - a_S\int_0^\infty e^{-pt}\,\lambda\, dt = 0$$

Integrating the first integral by parts twice

$$\int_0^\infty e^{-pt}\frac{d}{dt}\frac{S}{dt}\lambda\, dt = e^{-pt}\frac{d\lambda}{dt}\Big|_0^\infty + p\, e^{-pt}\lambda\Big|_0^\infty + p_S\int_0^\infty e^{-pt}\,\lambda\, dt,$$

notice that:

1. $e^{-pt}\dfrac{d\lambda}{dt}\Big|_0^\infty = -\dfrac{d\lambda}{dt}(0) = 0$ due to the initial conditions.

2. $pe^{-pt}\lambda\Big|_0^\infty = -p\lambda(0) = -p$

The equation may now be written in a factored form:

$$(p^2 - a^2) \int_0^\infty e^{-pt}\, y\, dt - p = 0$$

or:

$$\int_0^\infty e^{-pt}\, y\, dt = \frac{p}{p^2 - a^2}$$

This equation says that the LaPlace transform of $y = \frac{p}{p^2 - a^2}$.

Thus by consulting a table of LaPlace transforms you may readily determine what function has a transform $\frac{p}{p^2 - a^2}$. This function is the solution of the given differential equation.

$$y = \cosh at$$

Check:

$$\frac{d}{dt} (\cosh at) = a \sinh at$$

$$\frac{d}{dt} (a \sinh at) = a^2 \cosh at$$

Substituting this into the differential equation it is observed that the equation is identically satisfied.

Are the initial conditions satisfied?

1. $\cosh (0) = 1 = y(0)$
2. $a \sinh (0) = 0 = \frac{dy(0)}{dt}$

Thus this solution satisfies both the differential equation and the initial conditions.

Application of the LaPlace Transform (5)

$$m \frac{d^2x}{dt^2} + F(x) = 0$$

choose $F(x) = kx + bx^3$

thus $\frac{d^2x}{dt^2} + \frac{k}{m} x + \frac{b}{m} x^3 = 0$

where $\frac{k}{m} = w^2$ and $\frac{b}{m} = \alpha$

the equation would be:

$$\frac{d^2x}{dt^2} + w^2x + \alpha x^3 = 0.$$

Initial Conditions:
suppose, at $t = 0$ $\begin{cases} x = a \\ \frac{dx}{dt} = 0 \end{cases}$

Taking the Transform of the given equation

$$\int_0^\infty e^{-pt} \left[\frac{d^2x}{dt^2} + w^2x + \alpha x^3 \right] dt = 0$$

Using the properties of the transform as previously stated,

$$L\left\{\frac{d^2x}{dt^2}\right\} = p^2L\{x\} - p\, f(0) - \frac{d\, f(0)}{dt}$$

$x(0) \equiv f(0)$ thus

$$p\, f(0) = ap$$

$$\frac{d\, f(0)}{dt} = 0$$

let $\bar{x} \equiv L\{x\}$

$$L\{w^2x\} = w^2\bar{x}$$

The expression of the differential equation may now be put in the form:

$$p^2\bar{x} - pa + w^2\bar{x} + \alpha L\{x^3\} = 0$$

or

$$(p^2 \quad w^2)\ \bar{x} = pa - \overline{\alpha x^3}$$

Let $x = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots$

$w^2 = w_0^2 + \alpha c_1 + \alpha^2 c_2 + \alpha^3 c_3 + \ldots$

$\bar{x} = \bar{x}_0 + \alpha \bar{x}_1 + \alpha^2 \bar{x}_2 + \alpha^3 \bar{x}_3 + \ldots$

Here all powers of α higher than the 3rd shall be omitted.

$$\therefore (p^2 + w_0^2 + \alpha c_1 + \alpha^2 c_2 + \alpha^3 c_3)(\bar{x}_0 + \alpha \bar{x}_1 + \alpha^2 \bar{x}_2 + \alpha^3 \bar{x}_3) = pa - \alpha L\{(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3)^3\}$$

Performing the indicated operations on the above equation and grouping like powers of α gives:

$$\bar{x}_0 p^2 + w_0^2 \bar{x}_0 + \alpha\left[c_1 \bar{x}_0 + p^2 \bar{x}_1 + w_0^2 \bar{x}_1 + L\{x_0^3\}\right]$$
$$+ \alpha^2\left[p^2 \bar{x}_2 + w_0^2 \bar{x}_2 + c_2 \bar{x}_0 + c_1 \bar{x}_1 + 3L\{x_1 x_0^2\}\right]$$
$$+ \alpha^3\left[p^2 \bar{x}_3 + w_0^2 \bar{x}_3 + c_1 \bar{x}_2 + c_2 \bar{x}_1 + c_3 \bar{x}_0 + L\{3x_0 x_1^2 + 3x_2 x_0^2\}\right] = pa.$$

If this is to be an identity the coefficients of like powers of α may be equated on each side of the equation.

The first equation would then be

$$\bar{x}_0 p^2 + w_0^2 \bar{x}_0 = pa$$

There are two things that should be noted about this equation:

1. It completely determines x_0 which is the first approximation.

2. Further refinement of the solution will not change x_0. Thus with this method the complete solution is not required in order to determine whether sub harmonics are present.

$$\bar{x}_0 = \frac{ap}{p^2 + w_0^2}$$

$\therefore$ From a table of transforms,

$$\underline{x_0 = a \cos w_0 t}$$

This result carries the method as far as would be necessary to determine whether or not a sub harmonic was present.

Application of LaPlace Transform to a Forced System

$$\frac{d^2x}{dt^2} + w^2x + \alpha x^3 = A \sin 2w_0 t$$

Here the notation is consistent with the last section.

$$\int_0^\infty e^{-pt} \left[\frac{d^2x}{dt^2} + w^2x + \alpha x^3\right] dt = A \int_0^\infty e^{-pt} \sin 2wt \, dt$$

The left side of the equation will be the same as it was in the previous section.

$$\int_0^\infty e^{-pt} \left[\frac{d^2x}{dt^2} + w^2x + \alpha x^3\right] dt = (p^2 + w^2)\,\bar{x} - pa + \alpha L\{x^3\}$$

$$L\left\{ A \sin 2wt \right\} = \frac{2A\, w_0}{p^2 + 4w_0^2}$$

If we expand x, w, and $\bar{x}$ in series exactly as was done in the last section nothing will change except L A sin 2wt will be added to the right side of the equation.

Since only a first approximation is desired to determine whether or not a sub harmonic is present only the first term of the left side will be reproduced here.

$$p^2\bar{x}_0 + w_0^2\bar{x}_0 = pa + A\left[\frac{w_0}{p^2 + 4w_0^2} \right]$$

The w will contain ω_0 terms and as such will be necessary to obtain higher approximations but at least we will be assured of a term whose LaPlace transform is

$$\frac{pa}{p^2 + w_0^2} = L\left\{ a \cos w_0 t \right\}$$

Thus it has been shown that a subharmonic will be present in the solution of such an equation.

Application of the LaPlace Transform to the Case of an Unsymmetric Restoring Force.

$$m\frac{d^2x}{dt^2} + kx + ax^2 = A \sin 2w_0 t$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x + \frac{a}{m}x^2 = \frac{A}{m}\sin 2w_0 t$$

let $\quad \frac{k}{m} = w^2 \qquad \frac{a}{m} = \alpha \qquad \frac{A}{m} = F$

then $\quad \frac{d^2x}{dt^2} + w^2x + \alpha x^2 = F \sin 2w_0 t$

Initial Conditions: $\quad$ at $t = 0$ $\quad \begin{cases} x = a \\ \frac{dx}{dt} = 0 \end{cases}$

$$\int_0^\infty e^{-pt}\left[\frac{d^2x}{dt^2} + w^2x + \alpha x^2\right] dt = L\left\{F \sin 2w_0 t\right\}$$

$$L\left\{\frac{d^2x}{dt^2}\right\} = p^2\bar{x} - pa$$

$$L\left\{w^2x\right\} = w^2\bar{x}$$

$$L\left\{\alpha x^2\right\} = \alpha L\left\{x^2\right\}$$

The equation may now be written in the form :

$$p^2\bar{x} + w^2\bar{x} - pa + \alpha L\left\{x^2\right\} = L\left\{F \sin 2w_0 t\right\}$$

$$(p^2 + w^2)\,\bar{x} = pa + L\left\{F \sin 2w_0 t\right\} + \alpha L\left\{x^2\right\}$$

Assume $\alpha << 1$

Application of the Laplace Transform to the Case of an Unsymmetric Restoring Force.

Let

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \dots$$

$$w^2 = w_0^2 + \alpha c_1 + \alpha^2 c_2 + \alpha^3 c_3 + \dots$$

$$\bar{x} = \bar{x}_0 + \alpha \bar{x}_1 + \alpha^2 \bar{x}_2 + \alpha^3 \bar{x}_3 + \dots$$

Substituting these relationships into the equation:

$$\left[p^2 + w_0^2 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3\right]\left[\bar{x}_0 + \alpha \bar{x}_1 + \alpha^2 \bar{x}_2 + \alpha^3 \bar{x}_3\right] =$$

$$pa + L\left\{F \sin 2w_0 t\right\} + \alpha L\left\{x^2\right\} .$$

Since we are not necessarily interested in a complete solution of the equation but only in whether or not a subharmonic term is present in the solution we shall take only a first approximation.

$$(p^2 + w_0^2)\,\bar{x}_0 = pa + L\left\{F \sin 2w_0 t\right\}$$

$$x_0 = \frac{pa}{p^2 + w_0^2} + \frac{L\left\{F \sin 2w_0 t\right\}}{p^2 + w_0^2}$$

Considering only the first term,

$$\bar{x}_0 = \frac{pa}{p^2 + w_0^2}$$

$$\therefore \quad x_0 = a \cos w_0 t + L\left\{\frac{L\left\{F \sin 2w_0 t\right\}}{p^2 + w_0^2}\right\}.$$

This shows that the subharmonic frequency w_0 will be present in the solution.

ABSTRACT

ABSTRACT OF THESIS

What is subharmonic resonance? Where and why should it occur? These are some of the questions which are considered in this paper.

Subharmonic resonance is a phenomenon that occurs in many vibrating systems. It has been observed in fields as widely separated as acoustics and aerodynamics. F.R.W. Strafford observed second subharmonics in a moving coil loudspeaker, while others have observed airplane surfaces vibrating at a frequency which was a subharmonic of the engine frequency. Essentially, subharmonic resonance occurs whenever a system vibrates with a frequency $\frac{1}{n}$ times the driving frequency where n may be any positive integer.

At present there seems to be no clear cut explanation of this phenomenon. It has been suggested that higher harmonics occur in such profusion that among these harmonics there may be contained a series, which has, as its fundamental, a frequency lower than the driving frequency. Due to the non-linearity of the system this "fundamental" frequency may be excited. There is some basis for this since it is well known that the fundamental may be filtered out of a sound and that the ear, due to non-linearity, will fill in the fundamental.

Subharmonic resonance does not always arise because of

non-linearity. Take the case of a string, one end of which is fixed, and the other end attached to a tuning fork. The tuning fork may then be set into vibration and will drive the string. Since the string can undergo tension only, it is excited by the tuning fork only half the time. Thus the string will vibrate at half the frequency of the tuning fork.

The experiment which was adopted to establish the existence of subharmonic resonance gave absolutely no indication of this phenomenon. It has been suggested that one possible reason for the experiment's failure was that not enough power was put into the system to excite the subharmonic frequency to an observable level.

The second experiment was designed to eliminate this condition. The subharmonic frequency is generated by a non-linear inductance, is selected, if present, by a tank circuit, and is amplified until observable on the oscilloscope. Due to the lack of time this experiment was not performed.

The theoretical aspect of this paper deals with four different methods of solving the differential equations which arise in the consideration of certain non-linear vibrating systems.

The method which is developed to the greatest extent in this paper is due to Dr. P. O. Pedersen. He assumed small oscillations and small non-linear terms. The method then boils down to substituting a Fourier series into the given

differential equation and making certain simplifications based on the assumptions. The relationships which are then found enable him to arrive at an approximate solution. The most general form of the differential equation is considered first, and then it is specialized and simplified as convenience and interest dictate. It was observed that two rather difficult cases could be transformed into a simpler case, already considered, if the constants in the linear variation of mass, resistance, and stiffness were considered equal. This would not be expected in the most general case, but it is probable that such a condition could be produced in the laboratory.

The methods developed in the treatment of non-linear vibrating systems by using the LaPlace Transform are due to L.A. Pipes and may be found his book, "Applied Mathematics for Engineers and Physicists." This method assumes that only a small degree of non-linearity is present in the system. If "a" is a measure of the non-linearity, then the solution to the equation is developed in powers of "a". This can be seen to converge if "a" is small compared to unity . To obtain higher approximations using this method one has only to add more terms rather than making an essential change in the method.

The numerical method makes use of the fact that the derivative may be calculated when only numerical values of a function are given. Once these derivatives are known a

Taylor series may be formed. This is a solution to the equation, but it is not the desired form. From the Taylor series a Fourier series may be computed. This method would not be very practical unless the equation would yield to no other treatment.

The method used by T. von Karman employs an iterative procedure and also assumes a small non-linearity in the system. The first approximation is found to be periodic and is then immediately expanded into a Fourier series. This procedure is somewhat more general than others considered since an arbitrary function is maintained throughout.

It is interesting to notice that in every case considered a subharmonic term was found in the solution. This fact leads me to believe that subharmonics are present in all non-linear vibrating systems.

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FIRST READER'S REPORT

Author of Thesis: John E. Mulhern, Jr.

Title of Paper: Subharmonic Resonance

CRITICAL ESTIMATE OF FIRST DRAFT

This paper by Mr. Mulhern qualifies as a Master's Thesis. Mr. Mulhern has summarized a number of methods used in solving non-linear differential equations, and has extended their application to subharmonic resonance in some instances. This work is of interest and shows some originality. The paper is perhaps a little brief on explanation, and might be difficult to read by someone not familiar with the field. Any changes which may be needed will be discussed with Mr. Mulhern.

The thesis is approved
~~is/not~~

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GRADUATE SCHOOL
Boston 15, Massachusetts

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